More ways to tell if a series converges or diverges.

$\boxed{\text{Ex}}$ $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \infty$

harmonic.

$\boxed{\text{Ex}}$ $1+x+x^2+x^3+\cdots = \frac{1}{1-x}$ geometric.

converges if $|x|<1$

$\boxed{\text{Ex}}$ Does this converge?

$S_\infty = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ "Alternating harmonic series"

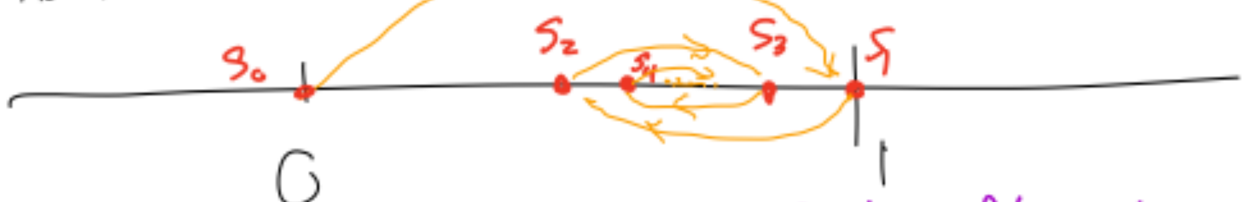
S_1

S_2

S_3 ← partial sums

i.e. Does $\lim_{n\to\infty} S_n$ exist?

Plot partial sums on real line



Actually, this series converges ⊙ signs alternate
⊙ the amount we are adding/subtracting decreases
⊙ that # → 0.

↑ This is an example of the alternating series test:

If a series $\sum a_k$ is ① alternating (i.e. signs alternate $+-+-,-+-+\ldots$)

② $|a_k|$ decreases with k

i.e. $|a_{k+1}| \le |a_k|$ for $k \ge 1$

③ $\lim\limits_{k\to\infty} a_k = 0 = \lim\limits_{k\to\infty} |a_k|$ — if one $=0$, the other limit $=0$

Then $\sum a_k$ converges.

(Ex) Does $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^{3/2}+2}$ converge?

Answer: ① Alternating $(-1)^k = \begin{cases} -1 & k=1 \\ +1 & k=2 \\ -1 & k=3 \\ \vdots \end{cases}$ ✓

② $\dfrac{k}{k^{3/2}+2}$ decreases as k increases (denom increases faster than numerator.) $3/2 > 1$ ✓

③ $\displaystyle\lim_{k\to\infty} |a_k| = \lim_{k\to\infty} \frac{k}{k^{3/2}+2} = \lim_{k\to\infty} \frac{k/k^{3/2}}{\frac{k^{3/2}}{k^{3/2}} + 2/k^{3/2}}$

$\displaystyle = \lim_{k\to\infty} \frac{\frac{1}{k^{1/2}} \to 0}{1 + \frac{2}{k^{3/2}} \to 0} = 0.$ ✓

∴ Series converges.

Two more tests that are derived from what we know about the geometric series.

$$\sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} ar^k \quad \text{converges if } |r|<1$$
$$\text{diverges if } |r| \geq 1.$$
$$\left(\text{goes to } \infty \text{ fast if } |r|>1\right)$$

$$b_k = a \cdot r^k$$

Note: $r = \frac{b_{k+1}}{b_k}$. $\left(\text{Why? } \frac{b_{k+1}}{b_k} = \frac{a \cdot r^{k+1}}{a \cdot r^k} = r\right)$

If a series has limiting behavior like the geometric series, $\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = r$.

From this, we obtain:

$\boxed{\text{Ratio Test}}$: Consider $\sum_{k=0} b_k$.

If $\lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| = L$ (exists), then

1. If $L<1$, $\sum b_k$ converges (absolutely) ($\sum |b_k|$ converges)
2. If $L>1$, $\sum b_k$ diverges
3. If $L=1$, test is inconclusive (sometimes it converges, sometimes it diverges)

Another test derived from geometric series:

geometric $\sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} a \cdot r^k$

$\lim_{k\to\infty} \sqrt[k]{|b_k|} = \lim_{k\to\infty} \sqrt[k]{|a \cdot r^k|}$

$|AB| = |A|\,|B|$

$= \lim_{k\to\infty} \left(|a|\,|r|^k\right)^{1/k} = \lim_{k\to\infty} \underbrace{|a|^{1/k}}_{\to 1} |r| = |r|.$

$\Rightarrow$ $\boxed{\text{Root Test}}$

Consider $\sum_{k=0}^{\infty} b_k$. If $\lim_{k\to\infty} |b_k|^{1/k} = L$ exists, then

1. If $L < 1$, series $\sum b_k$ converges absolutely (i.e. $\sum |b_k|$ also converges)
2. If $L > 1$, series $\sum b_k$ diverges.
3. If $L = 1$, then the test is inconclusive (sometimes converges, sometimes diverges.)

Quiz.

1. Taylor series formula for $f(x)$ @ $x=a$

   $T_\infty(x) =$ ______

2. Taylor series for $e^x =$ ______

3. Taylor series for $\cos(x) =$ ______

4. What is your name?

5. Favorite book = ______

Another Taylor Series for notes

$$\boxed{(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots}$$

Binomial series